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NOTE

On the Direct Summation of Series Involving Higher Transcendental Functions*

I. INTRODUCTION

In many problems of physics, there is often the need of evaluating or computing a series of the form

$$S_N(x) = \sum_{n=j}^{N} a_n(x) f_n(x) \tag{1}$$

where $f_n(x)$ is a higher transcendental function and $a_n(x)$ are given coefficients. Well known examples of such are truncated series involving Chebychev polynomials, Bessel functions, and Legendre functions. In most applications we have $j = 0$ or 1. Many higher transcendental functions satisfy a three-term recurrence ralation of the form

$$f_{n+1}(x) = B_n(x) f_n(x) + C_n(x) f_{n-1}(x) \qquad n = 0, 1,... \tag{2}$$

In Table I we summarize some common recurrences. For the orthogonal polynominals, we define $f_{-1}(x) = 0$.

It is well known that recurrence relations form a basic mathematical tool for the computation of many functions. We have, for example, Miller's algorithm for computing Bessel functions. For a recent detailed survey and analysis of such algorithms, the reader is referred to Gautschi [1]. Whereas these relations are simple to use, one must attend to the problem of numerical stability. For example, Gautschi shows that given $J_0(1)$ and $J_1(1)$ accurate to 10 significant figures and generating the next values of $J_n(1)$ by forward recursion, one loses all significance for $n \geqslant 7$. Abramowitz [2] summarizes the caution one must take in using such recurrence relations. In particular, the direction of recurrence is important. For example, the Bessel functions J_n and I_n are stable only in backward recurrence whereas Y_n and K_n are stable only in forward recurrences.

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TABLE I

Coefficients for Recurrence Relations

FUNCTIONS	f_n	B_n	C_n
Chebychev Polynomials	T_n	$2x$	-1
Legendre Polynomials	P_n	$\frac{2n+1}{n+1}x$	$-\frac{n}{n+1}$
Legendre Functions	$P_l^{\,n}$ $P_n^{\,m}$	$-2nx(1-x^2)^{-1/2}$ $\frac{2n+1}{n+1-m}x$	$(n+l)(n-l-1)$ $\frac{m+n}{m-n-1}$
Hermite Polynomials	H_n	$2x$	$-2n$
Laguerre Polynomials	L_n	$\frac{2n+1-x}{n+1}$	$-\frac{n}{n+1}$
Bessel Functions	$J_n,\ Y_n,\ H_n^{(1)},\ H_n^{(2)}$	$\frac{2n}{x}$	-1
Modified Bessel Functions	I_n K_n	$\mp\frac{2n}{x}$	1
Coulomb Wave Functions	$\left.\begin{matrix}F_n(\eta,\rho)\\G_n(\eta,\rho)\end{matrix}\right\}$	$\frac{2n+1}{n}\left[\eta+\frac{n(n+1)}{\rho}\right][(n+1)^2+\eta^2]^{-1/2}$	$-\frac{n+1}{n}[n^2+\eta^2]^{1/2}[(n+1)^2+\eta^2]^{-1/2}$

II. An Algorithm for the Direct Summation of $S_N(x)$

Clenshaw [3] recommends an algorithm to sum a Chebychev series directly. In this section we shall generalize the algorithm to other special functions satisfying Eq. (2).

Consider the recurrence formula (with the functional dependence on x understood),

$$b_k = b_{k+1}B_k + b_{k+2}C_{k+1} + a_k ,$$
$$b_{N+1} = b_{N+2} = 0, \qquad k = N, N-1,..., j. \tag{3}$$

Multiply Eq. (3) by f_k and write down a "system of equations" as follows:

$$\begin{aligned} b_N f_N &= && && + a_N f_N \\ b_{N-1} f_{N-1} &= b_N f_{N-1} B_{N-1} && && + a_{N-1} f_{N-1} \\ b_{N-2} f_{N-2} &= b_{N-1} f_{N-2} B_{N-2} &&+ b_N f_{N-2} C_{N-1} &&+ a_{N-2} f_{N-2} \\ &\vdots \\ b_j f_j &= b_{j+1} f_j B_j + b_{j+2} f_j C_{j+1} + a_j f_j \end{aligned} \tag{4}$$

Adding up all equations of (4) and using Eq. (2), we obtain,

$$S_N = \sum_{n=j}^{N} a_n f_n = b_j f_j + b_{j+1}(f_{j+1} - B_j f_j). \tag{5}$$

Notice that Eq. (3) is a backward recurrence, but not as the nonhomogeneous counterpart of Eq. (2) because the role of C_k is displaced. Obviously one can also derive a recurrence scheme expressing S_N in terms of f_N and f_{N-1}. Notice that for $j = 0, S_N = b_0$ for the orthogonal polynomials. Thus Eq. (3) represents a formalism for computing the series S_N. It is mainly useful for the case $j = 0$ or 1 because here f_0 and f_1 are readily obtainable. But the applicability will of course depend on the stability of Eq. (3), which in turn depends on the function in question. In the following, we shall describe some numerical experiments with this algorithm, by applying it to the following simple series (Mangulis, [4]):

$$J_0(\pi) + 2 \sum_{n=1}^{\infty} (-1)^n J_{2n}(\pi)\, T_{2n}(x) = \cos \pi x, \qquad -1 \leqslant x \leqslant 1. \tag{6}$$

$$I_0(1) + 2 \sum_{n=1}^{\infty} (-1)^n I_n(1)\, T_n(x) = e^x, \qquad -1 \leqslant x \leqslant 1. \tag{7}$$

$$\sum_{n=0}^{\infty} (2n+1)\, z^n P_n(x) = \frac{1 - z^2}{(1 - 2xz + z^2)^{3/2}}, \qquad -1 \leqslant x \leqslant 1, |z| \leqslant 0.6 \tag{8}$$

$$\sum_{n=1}^{\infty} \frac{\alpha^n P_n(x)}{n+1} = \frac{1}{\alpha} \log \left[\frac{\alpha - x + (1 - 2\alpha x + \alpha^2)^{1/2}}{1 - x}\right] \quad |\alpha| < 1 \qquad (9)^1$$

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} P_n(x) = e^{xz} J_0[z(1 - x^2)^{1/2}], \qquad -1 \leqslant x \leqslant 1, |z| \leqslant 4. \qquad (10)$$

$$\sum_{n=0}^{N} A_n P_n(x), \qquad -1 \leqslant x \leqslant 1, N = 20, 30, 40, 0 \leqslant A_n \leqslant 1, \text{ and } 0 \leqslant A_n \leqslant 100. \qquad (11)$$

$$\sum_{n=0}^{N} A_n L_n(x), \qquad 0 \leqslant x \leqslant 100, N = 20, 30, 40, 0 \leqslant A_n \leqslant 100. \qquad (12)$$

All computations were performed on an IBM 7094 using double precision (16 decimal digit) arithmetic. For Eqs. (6) to (10), we terminate the series when the coefficient is less than 10^{-17}. For each series we generate 1000 uniformly distributed pseudorandom numbers for the variables x and z in the indicated range, which does not necessarily cover the whole range of theoretical convergence. The choice of range is obviously for practicality. For example, for Eq. (8), at $|z| = 0.6$ one needs about 100 terms to satisfy our criterion. In Eqs. (11) and (12) the A_n's are a set of pseudorandom numbers uniformly distributed in the indicated range. In all of the above series, we also compute the sum by generating the special functions by forward recurrence and then summing. Thus we have 3 different results for Eqs. (6) to (10) and 2 for Eqs. (11) and (12). In all cases we compute the relative differences among the 2 or 3 different methods. These differences of course depend on the values of x, z, A_n, S_N and N. They range from 1×10^{-16} to 1×10^{-14}, but are in no case greater than the last number.

The algorithm described in this note is in principle applicable to the computation of double series. For example, consider the series of Legendre functions,

$$S_N = \sum_{l=2}^{N} \sum_{n=0}^{l} A_{nl} P_l^n \equiv \sum_{l=2}^{N} s_l, \qquad (13)$$

where again the functional dependence on x is understood. The inner series can be summed by the recursion

$$b_l^n = -2nx(1 - x^2)^{-1/2} b_l^{n+1} + (n + l + 1)(n - l) b_l^{n+2} + A_{nl} \qquad (14)$$

$$b_l^{l+1} = b_l^{l+2} = 0, \qquad n = l, l - 1, \ldots, 0, \qquad l = 2, 3, \ldots, N.$$

[1] See Magnus, Oberhettiger, and Soni, *Formulas and Theorems of Math-Physics*, Band 52, Springer, Verlag, N.Y., 1966.

We obtain,

$$s_l = b_l^0 P_l + b_l^1 P_l^1.$$

By reducing P_l^1 to a combination of P_l and P_{l-1}, we can again obtain S_N as a series involving P_l. It is evident that in applying Eq. (14) one needs to exert caution in scaling to avoid overflow.

III. Acknowledgment

After the original manuscript was written, Mr. W. J. Cody called my attention to a similar account of this algorithm by Professor J. Rice (Hart, *et al.* [5]). Thus this paper should be considered expository in character. I am also indebted to Dr. C. L. Lawson and Dr. M. Saffren for helpful discussions, and to the referee for helpful comments.

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